Dynamics of the Macpherson Strut Motor-Vehicle Suspension System in Point and Joint Coordinates

Hazem Ali Attia*

Department of Mathematics, College of Science, King Saud University (Al-Qasseem Branch),
P.O. Box 237, Buraidah 81999, Kingdom of Saudi Arabia

In this paper the dynamic analysis of the Macpherson strut motor-vehicle suspension system is presented. The equations of motion are formulated using a two-step transformation. Initially, the equations of motion are derived for a dynamically equivalent constrained system of particles that replaces the rigid bodies by applying Newton's second law. The equations of motion are then transformed to a reduced set in terms of the relative joint variables. Use of both Cartesian and joint variables produces an efficient set of equations without loss of generality. For open chains, this process automatically eliminates all of the non-working constraint forces and leads to an efficient solution and integration of the equations of motion. For closed loops, suitable joints should be cut and few cut-joints constraint equations should be included for each closed chain. The chosen suspension includes open and closed loops with quarter-car model. The results of the simulation indicate the simplicity and generality of the dynamic formulation.

Key Words : Dynamic analysis, Suspension System, Vehicle Simulation, Equations of Motion, Macpherson Strut

1. Introduction

Different formulations are used to model and simulate the dynamics of machines and mechanisms. Computer programs are designed to generate and integrate the governing equations of motion for a system of rigid bodies. Some of these programs generate the equations of motion in the form of a large set of coupled differential-algebraic equations (Orlandea et al., 1978 ; Serna et al., 1982 ; Nikravesh, 1988) while others generate a minimal set of differential-algebraic equations of motion (Kim and Vanderploeg, 1985 ; 1986).

One formulation which can yield a minimum number of differential equations of motion is

based upon both point and joint coordinates (Attia, 1993, 1996, 1998 ; Nikravesh and Attia, 1994 ; Attia and Mohamed, 1997). The formulation initially uses the rectangular Cartesian coordinates of an equivalent constrained system of particles to define the configuration of the system. The differential equations of motion are derived by applying Newton's second law to study the translational motion of the particles. The resulting differential equations of motion together with the equations of constraints constitute a mixed set of differential-algebraic equations that can be solved at every time step to determine the global motion of the system. For the purpose of computational efficiency, the equations of motion derived in the point coordinates are then transformed to a reduced set of equations in terms of a selected set of relative joint coordinates. This transformation is done using a velocity transformation which relates the Cartesian velocities of the particles to the relative joint velocities and allows for an efficient solution and integration of the equations of motion.

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* **E-mail :** ah1113@yahoo.com
 TEL : +966-6-3800319; **FAX :** +966-6-3800911
 Department of Mathematics, College of Science, King Saud University (Al-Qasseem Branch), P.O. Box 237, Buraidah 81999, Kingdom of Saudi Arabia. (Manuscript **Received** October 30, 2002; **Revised** June 3, 2003)

This paper briefly describes the point and joint coordinate formulation. Then the formulation is applied to study the dynamic analysis of the MacPherson strut suspension system. The model includes a quarter car and the suspension together with the necessary elements of the coil springs and shock absorbers. The tire is modelled simply by a spring-damper element. The results of the simulations are tested and compared with DAP-3D program which is based on the absolute coordinate formulation (Nikravesh, 1988). Results from some selected simulations are presented which indicate the simplicity and generality of the dynamic formulation.

2. Equations of Motion of a System of Rigid Bodies

2.1 Equations of motion in the point coordinates

A rigid body can be represented by a dynamically equivalent constrained system of constrained particles. The system of particles is constructed such that it has the same mass, position of the centre of mass and inertia components as its corresponding rigid body. Hence, in order to specify the position of the rigid body with respect to a non-moving coordinate system, it is sufficient to specify the position of its equivalent system of particles. There is a minimum number of particles for such an equivalence to be fulfiled, named as primary particles, and extra particles, called secondary particles, may be added also to achieve additional requirements (Attia, 1993 ; Nikravesh and Attia, 1994). According to the geometry and the type of motion under consideration, the rigid body may be represented by an equivalent system of primary particles using the two or three or four-primary-particles representations. Between these particles, there exist geometric constraints of the type,

$$(\mathbf{r}_i - \mathbf{r}_j)^T (\mathbf{r}_i - \mathbf{r}_j) - d_{i,j}^2 = 0$$

Such a constraint keeps the distance $d_{i,j}$ between particles i and j on the same rigid body constant. The external forces and couples acting on the system are known forces either constant such as

gravitational forces or variable such as spring or damper forces. These external forces or couples are transformed to an equivalent system of forces and distributed over the primary particles.

For a constrained system, kinematic joints between the rigid bodies are imposing kinematic constraints on the relative motion between the adjacent rigid bodies. These constraints can be expressed in the form of algebraic constraints between the primary particles (Attia, 1993 ; Nikravesh and Attia, 1994). For example if the primary particle i on body k coincides with the primary particle j on body l (e.g., a spherical joint) then we write a vector constraint equation in the form,

$$\mathbf{r}_i^k - \mathbf{r}_j^l = 0$$

However, by allowing bodies to share primary particles these simple kinematic constraints can be automatically eliminated and then reducing the total number of primary particles. Consequently, the mass of the primary particle, which is shared by the two bodies, receives contribution from both bodies and also the applied force on this particle receives contribution from the forces that act on both bodies.

The equations of motion of the equivalent system of particles are derived by applying Newton's second law to study the motion of the individual particles. The translational motion of the particles together with the constraints imposed on them define the general motion of the rigid body, both translational and rotational. The equations of motion of the secondary particles, which are located at the mid-points of the lines joining the primary particles, can be eliminated by expressing their unknown accelerations in terms of those of the primary particles with the aid of geometric constraints (Attia, 1993 ; Nikravesh and Attia, 1994). If the whole mechanical system is replaced by N primary particles, then the vector of Cartesian coordinates and the corresponding vectors of velocities and accelerations each has 3 N components. If the geometric and kinematic constraints contribute to m independent holonomic constraints then, the resulting equations of motion of the system of N constrained primary particles

consist of $3N+m$ differential-algebraic equations.

The constraint equations, both geometric and kinematic of the holonomic type, can be put in the general form,

$$\phi(\mathbf{q}) = 0$$

the first and second time derivatives of the constraints yield respectively to the velocity and acceleration equations of constraints in the form,

$$\dot{\phi}(\mathbf{q}) = \mathbf{D}\dot{\mathbf{q}} = 0$$

$$\ddot{\phi}(\mathbf{q}) = \mathbf{D}\ddot{\mathbf{q}} - \gamma = 0$$

where $\mathbf{D} = \partial\phi/\partial\mathbf{r}$ is the Jacobian matrix of the constraints $(m \times 3N)$ and γ is the right-hand-side of the acceleration equations of constraints $(m \times 1)$;

$$\gamma = -\dot{\mathbf{D}}\dot{\mathbf{q}}$$

Each particle is being acted upon by a known external force and an unknown constraint forces associated with the geometric and kinematic constraints. The unknown constraint forces can be expressed in terms of Lagrange multipliers in the form (Nikravesh, 1988),

$$\mathbf{f}_c = -\mathbf{D}^T \lambda$$

where $\mathbf{f}_c$ is the vector of unknown constraint forces $(3N \times 1)$ and λ is the vector of Lagrange multipliers associated with the geometric and kinematic constraints $(m \times 1)$. The final form of the equations of motion is given by (Nikravesh and Attia, 1994),

$$\begin{bmatrix} \mathbf{M} & -\mathbf{D}^T \\ \mathbf{D} & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{g} \\ \gamma \end{bmatrix} \qquad (1)$$

where $\mathbf{M}$ is the overall mass matrix $(3N \times 3N)$, $\ddot{\mathbf{q}}$ is the vector of Cartesian accelerations of the primary particles $(3N \times 1)$ and $\mathbf{g}$ is the vector of external forces $(3N \times 1)$; $\mathbf{g} = [\mathbf{g}_1, \cdots, \mathbf{g}_N]^T$ and where $\mathbf{g}_i$ is the resultant external force acting on particle i (3×1).

The use of particle dynamics expresses the general motion of the rigid body in terms of the translational motion of the particles and results in the elimination of the rotational equations of motion and the associated rotation matrices and leads to a simplified approach. The methodology

can be applied to either serial or parallel chains. The main disadvantage is the large resulting number of equations which can be greatly improved by the transformation to the relative joint variables.

2.2 Equations of motion in the joint coordinates

For multibody systems with open kinematic loops, Eq. (1) can be converted to a set of differential equations equals the number of degrees of freedom of the system using the velocity transformation approach. In the process of transformation, the position of a body is defined with respect to its adjacent reference body by relative angles or distances. Therefore, the vector of joint coordinates and the number of degrees of freedom are determined by the type of joints. If the relative configuration of two adjacent bodies is defined by relative joint coordinates equal in number to the number of relative degrees of freedom between the bodies, then a vector of relative joint coordinates is defined by $\boldsymbol{\theta} = [\theta_1, \cdots, \theta_n]^T$ where n is the number of degrees of freedom of the system. The chosen relative joint coordinates depend on the type of kinematic joints connecting the bodies. The corresponding vectors of relative joint velocities and accelerations can be derived by taking the first and second time derivatives of the relative joint coordinates and are respectively given by $\dot{\boldsymbol{\theta}} = [\dot{\theta}_1, \cdots, \dot{\theta}_n]^T$ and $\ddot{\boldsymbol{\theta}} = [\ddot{\theta}_1, \cdots, \ddot{\theta}_n]^T$. The absolute velocities of the particles belonging to one body depend on the absolute velocities of the preceeding body on the chain as well as the relative velocity between the two bodies. Such a relative velocity, is the one which can be expressed in terms of the relative joint velocities at the kinematic joints connecting the adjacent bodies. Thus, knowing the absolute velocities of the floating base body in a chain and the relative joint velocities we can sequentially determine the absolute velocities of the coming bodies on the chain. Finally, the vector of the Cartesian velocities of the particles can be described kinematically in terms of the vector of relative joint velocities and such kinematic relations are expressed as (Kim and Vanderploeg,

1986 ; Attia, 1993 ; Nikravesh and Attia, 1994),

$$\dot{q} = B\dot{\theta} \qquad (2)$$

where matrix B is called the velocity transformation matrix which is orthogonal to the Jacobian matrix D. Matrix B is derived using the kinematic relations for each joint type (Nikravesh and Attia, 1994). A systematic construction of the velocity transformation matrix B for open or reduced open loop systems is given in (Kim and Vanderploeg, 1986 ; Attia, 1993 ; Nikravesh and Attia, 1994). The structure of this matrix shows that it can be partitioned into submatrices (block matrices), which are associated with different types of kinematic joints. Then, each joint type has its corresponding block matrix which leads to a systematic construction of the matrix B. Substitution of the time derivative of Eq. (2) in Eq. (1) and premultiplication by B^T yield

$$\bar{M}\ddot{\theta} = f \qquad (3)$$

where $\bar{M} = B^T MB$ and $f = B^T(g - M\dot{B}\dot{\theta})$. It should be pointed out that the velocity transformation matrix B is orthogonal to the Jacobian matrix of the constraint equations (Attia, 1993 ; Nikravesh and Attia, 1994), then, the term $B^T f_c$ vanishes, where f_c are the constraint forces. Therefore, all the geometric and kinematic constraints are eliminated through the transformation to joint variables. The symmetric linear system of Eq. (3) represents the equations of motion for an open loop system when the number of selected relative joint coordinates is equal to the number of degrees of freedom. At every instant, knowing the external forces and the joint coordinates and velocities, Eq. (3) can be solved for the unknown joint accelerations.

To derive the equations of motion for systems containing closed loops, each closed loop is cut at one of the kinematic joints in order to obtain a reduced open loop system. For this reduced system, joint coordinates are defined as for any open loop system. If this system is closed at the cut joint(s), the joint coordinates will no longer be independent. Additional kinematic constraints due to cut-joints should be appended to the geometric and kinematic constraints associated

with the open loop constraints. The resulting constraint equations for the closed kinematic loops may be expressed as

$$\psi(\theta) = 0 \qquad (4a)$$

The time derivative of the constraints are

$$\dot{\psi} = C\dot{\theta} = 0 \qquad (4b)$$

$$\ddot{\psi} = C\ddot{\theta} + \dot{C}\dot{\theta} = 0 \qquad (4c)$$

where C is the Jacobian matrix of the cut joints constraints as well as the original constraints of the open loop system. Then, the equations of motion, Eq. (3), are modified for closed loop systems as (Attia, 1993 ; Nikravesh and Attia, 1994),

$$\begin{bmatrix} \bar{M} & -C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ v \end{bmatrix} = \begin{bmatrix} f \\ -\dot{C}\dot{\theta} \end{bmatrix} \qquad (5)$$

where v is the vector of Lagrange multipliers associated with the constraints of Eq. (4). Equations (5) represent the equations of motion for a multibody system when the number of selected joint coordinates is greater than the number of degrees of freedom of the system. At every instant, knowing the external forces, the joint coordinates and the joint velocities, Eq. (5) can be solved for the unknown joint accelerations and the cut-joint constraints forces.

The linear system of algebraic Eqs. (3) or (5) is solved using the Gaussian elimination technique adopted for symmetric matrices. Then, the joint accelerations are integrated twice to determine the joint coordinates and velocities at the next time step.

2.3 Integration of the equations of motion

The differential equations of motion for open loop system, Eq. (3), or for closed loop system, Eq. (5), represent a set of non-linear ordinary differential equations with the time as an independent variable that can be put in the standard form,

$$\dot{y} = h(y, t)$$

where y and $\dot{y}$ are vectors that contain the relative joint coordinates, velocities, and accelerations as

$$\dot{y} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} \text{ and } y = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

The numerical solution of the equations of motion requires a numerical integration process that determines the elements of **y** at every time step. The function "h" is evaluated by solving the equations of motion for the unknown joint accelerations. This numerical process is summarized as follows:

(1) Initially, the joint coordinates and velocities are known, i.e. **y**.

(2) Using the vector **y**, the Cartesian coordinates and velocities and the matrix **B**, **Ḃ**, **C** and **Ċ** can be constructed.

(3) With the knowledge of the known mass matrix **M** and the force vector **f**, the equations of motion for open loop system, Eq. (3), or closed loop system, Eq. (5) can be derived.

(4) Solve the equations of motion for **θ̈** and **v** using the Gaussian elimination technique adopted for symmetric matrices.

(5) Construct the **ẏ** vector and return the contents to the integration algorithm.

(6) repeat the previous steps at every time step.

Gear's method (Gear, 1988) for the numerical integration of differential-algebraic equations is used to overcome the instability problem resulting during the modelling process of constrained mechanical systems. The coordinate transformation from the joint space to the Cartesian space is done at every time step in order to construct the matrices **B**, **Ḃ**, **C**, and **Ċ**. Use of both Cartesian and joint coordinates produces an efficient set of equations without loss of generality.

3. Dynamic Simulation of the Macpherson Strut Suspension

In the past ten years much attention has been focused on improving the ride/handling compromise of the car by using a multi loop suspension and steering mechanism. The multi-loop structure usually gives the possibility to separate the wheel bouncing parameters determining ride comfort from steering. Figure 1(a) illustrates the multi-loop MacPherson strut suspension system mounted on the left side of the vehicle. The system has four degrees-of-freedom (DOF). The chassis

has one DOF since it is constrained to move vertically upward and downward which can be modelled as a translational joint with axis vertical. The wheel has one DOF corresponding to the
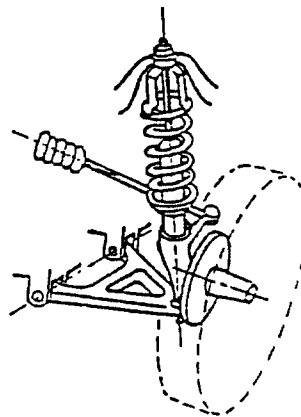

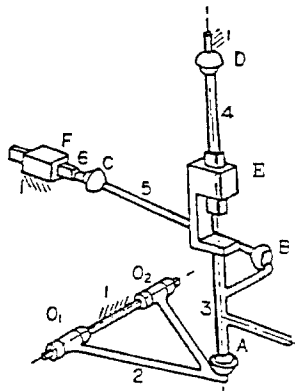**Fig. 1(a)**   The Macpherson strut suspension system

Fig. 1(b)   The Macpherson strut mechanism indicating body numbers
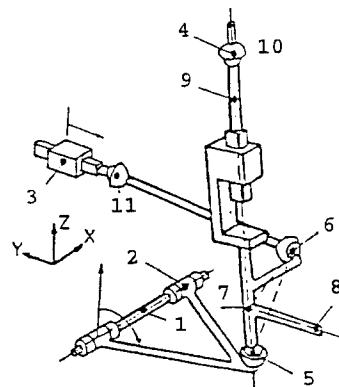

Fig. 1(c)   The Macpherson strut with the equivalent particles

rolling motion and is analytically modelled as a linear translational spring with damping characteristics. The MacPherson strut suspension mechanism has two DOF (see Fig. 1(b)). According to its performance, the mechanism can be separated into two independent parts; bouncing mechanism and steering mechanism. The bouncing mechanism is a one DOF four-bar linkage OAED with links 1, 2, 3, and 4. The bouncing action caused by the rotation of the lower arm (link 2) about the axis $O_1 O_2$ and the accompanying relative sliding motion of the portions of the strut (link 3 and 4). The lower portion of the strut is the wheel knuckle. The steering mechanism is a one DOF four-bar linkage FCBD. The steering action caused by the sliding motion of the steering rack (link 6), and accompanying motion of the tie rod (link 5) causing the rotation of the strut about the steering axis AD. Moreover, the rotation of the tie rod about its axis BC is a passive degree of freedom. Joints at A, B, C and D are spherical joints, while the coaxial joints at O_1 and O_2 are revolutes forming a compound revolute joint. The strut joint at E is a sliding joint with compliance. Thus, the two-DOF MacPherson strut suspension mechanism consists of six links, and four spherical joints, one compound revolute joint and two sliding joints. The MacPherson strut is being used for front wheel axles of current small cars, and can also be used for rear axles. The light weight and compact size of the mechanism are its main advantages. Furthermore, the system design allows longer axle springs, and thus a soft, long-strike suspension. A suspension spring and a shock absorber are included in the suspension sub-system. The inertia characteristics of the rigid bodies are presented in Table A.1. The characteristics of the suspension springs and dampers, and the wheel are presented, respectively, in Tables A.2 and A.3.

The whole system is replaced by its equivalent constrained system of particles. The chassis and the knuckle are replaced by the four-primary-particles representation. The A-arm is replaced by the three-primary-particles representation. The strut link 4, the tie rod (link 5) and the steering rack (link 6) each is replaced by the two-

primary-particles representation. The wheel is modelled with a system of two-primary-particles along its axis and a rotational variable. The whole system is represented by a resultant system of 11 particles. Figure 1(c) presents a schematic diagram for the mechanical system and the corresponding equivalent system of particles. Cutting the spherical joint at point D and the spherical-spherical joint along the tie rod (link 5) result in an open loop system that consists of two loops. Cut-joint constraint equations are required to locate particles 4 and 10 together and a fourth scalar equation is required to fix the distance between particles 6 and 11 along the tie rod. The vector of coordinates constitutes 3×11 Cartesian coordinates for the particles and one rotational variable for the rolling of the wheel, defined by,

$$\mathbf{q} = [\mathbf{r}_1^T, \cdots, \mathbf{r}_{11}^T, \theta_5]^T$$

where $\mathbf{r}_1, \cdots, \mathbf{r}_{11}$ are the global coordinates of the particles and θ_5 is the rolling angle of the wheel. The corresponding vectors of Cartesian velocities and accelerations are respectively given by,

$$\dot{\mathbf{q}} = [\dot{\mathbf{r}}_1^T, \cdots, \dot{\mathbf{r}}_{11}^T, \dot{\theta}_5]^T$$

$$\ddot{\mathbf{q}} = [\ddot{\mathbf{r}}_1^T, \cdots, \ddot{\mathbf{r}}_{11}^T, \ddot{\theta}_5]^T$$

The equations of motion in terms of the Cartesian coordinates are given by Eq. (3). The overall mass matrix is a 34×34 constant sparse symmetric matrix.

For the purpose of computational efficiency, the equations of motion derived in terms of the Cartesian coordinates of the particles are transformed to the relative joint variables. The vector of joint coordinates is defined by $\boldsymbol{\theta} = [\theta_1, \theta_2, \mathbf{p}_1^T, \theta_3, \theta_4, \theta_5]^T$. θ_1 is the vertical displacement of the chassis relative to a fixed coordinate frame, θ_2 is the rotation angle of the A-arm relative to the chassis, $\mathbf{p}_1$ are the Euler parameters that describe the orientation of the knuckle, θ_3 is the relative displacement between the strut links 3 and 4. θ_4 measures the sliding motion of link 6 and θ_5 is the rolling angle of the wheel. The corresponding vectors of joint velocities and accelerations are respectively given by,

$$\dot{\boldsymbol{\theta}} = [\dot{\theta}_1, \ \dot{\theta}_2, \ \omega_1^T, \ \dot{\theta}_3, \ \dot{\theta}_4, \ \dot{\theta}_5]^T \text{ and}$$

$$\ddot{\boldsymbol{\theta}} = [\ddot{\theta}_1, \ \ddot{\theta}_2, \ \omega_1^T, \ \ddot{\theta}_3, \ \ddot{\theta}_4, \ \ddot{\theta}_5]^T$$

$\dot{\boldsymbol{\omega}}_1$ and $\boldsymbol{\omega}_1$ are the angular velocity and acceleration of the knuckle respectively. The 34×8 velocity transformation matrix is constructed using the kinematic relations for every joint (Attia, 1993; Nikravesh and Attia, 1994). The details of the derivation of the velocity transformation equations are given in Appendix B and consequently the velocity transformation equation takes the form,

$$
\begin{bmatrix} \dot{\mathbf{r}}_1 \\ \dot{\mathbf{r}}_2 \\ \dot{\mathbf{r}}_3 \\ \dot{\mathbf{r}}_4 \\ \dot{\mathbf{r}}_5 \\ \dot{\mathbf{r}}_6 \\ \dot{\mathbf{r}}_7 \\ \dot{\mathbf{r}}_8 \\ \dot{\mathbf{r}}_9 \\ \dot{\mathbf{r}}_{10} \\ \dot{\mathbf{r}}_{11} \\ \dot{\theta}_5 \end{bmatrix} =
\begin{bmatrix}
\mathbf{u}_1 & & & & & \\
\mathbf{u}_1 & & & & & \\
\mathbf{u}_1 & & & & & \\
\mathbf{u}_1 & & & & & \\
\mathbf{u}_1 & -\tilde{\mathbf{d}}_{5,1}\mathbf{u}_2 & & & & \\
\mathbf{u}_1 & -\tilde{\mathbf{d}}_{6,1}\mathbf{u}_2 & -\tilde{\mathbf{d}}_{6,5} & & & \\
\mathbf{u}_1 & -\tilde{\mathbf{d}}_{7,1}\mathbf{u}_2 & -\tilde{\mathbf{d}}_{7,5} & & & \\
\mathbf{u}_1 & -\tilde{\mathbf{d}}_{8,1}\mathbf{u}_2 & -\tilde{\mathbf{d}}_{8,5} & & & \\
\mathbf{u}_1 & -\tilde{\mathbf{d}}_{9,1}\mathbf{u}_2 & -\tilde{\mathbf{d}}_{9,5} & \mathbf{u}_3 & & \\
\mathbf{u}_1 & -\tilde{\mathbf{d}}_{10,1}\mathbf{u}_2 & -\tilde{\mathbf{d}}_{10,5} & \mathbf{u}_3 & & \\
\mathbf{u}_1 & & & & & \mathbf{u}_4 \\
& & & & & 1
\end{bmatrix}
\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \omega_1 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \\ \dot{\theta}_5 \end{bmatrix}
$$

where $d_{i,j}$ is the 3×3 skew-symmetric tensors providing the vector product and $\mathbf{u}_i$ is the unit vector along the axis of the translational joint i. The corresponding 34×8 $\dot{\mathbf{B}}$ matrix has the form,

$$
\begin{bmatrix}
0 & & & \\
0 & & & \\
0 & & & \\
0 & & & \\
0 & -\tilde{\dot{\mathbf{d}}}_{5,1}\mathbf{u}_2 & & \\
0 & -\tilde{\dot{\mathbf{d}}}_{6,1}\mathbf{u}_2 & -\tilde{\dot{\mathbf{d}}}_{6,5} & \\
0 & -\tilde{\dot{\mathbf{d}}}_{7,1}\mathbf{u}_2 & -\tilde{\dot{\mathbf{d}}}_{7,5} & \\
0 & -\tilde{\dot{\mathbf{d}}}_{8,1}\mathbf{u}_2 & -\tilde{\dot{\mathbf{d}}}_{8,5} & \\
0 & -\tilde{\dot{\mathbf{d}}}_{9,1}\mathbf{u}_2 & -\tilde{\dot{\mathbf{d}}}_{9,5} & \bar{\mathbf{u}}_3(\dot{\theta}_2\mathbf{u}_2+\tilde{\omega}_1) \\
0 & -\tilde{\dot{\mathbf{d}}}_{10,1}\mathbf{u}_2 & -\tilde{\dot{\mathbf{d}}}_{10,5} & \bar{\mathbf{u}}_3(\dot{\theta}_2\mathbf{u}_2+\tilde{\omega}_1) \\
0 & & & 0 \\
& & & 0
\end{bmatrix}
$$

The cut-joint constraint equations are expressed in terms of the Cartesian coordinates of the particles. For particles 4 and 10, the constraint equation has the form,

$$\mathbf{r}_4 - \mathbf{r}_{10} = 0$$

For particles 6 and 11, the distance constraint has the form,

$$(\mathbf{r}_6 - \mathbf{r}_{11})^T (\mathbf{r}_6 - \mathbf{r}_{11}) - d_{6,11}^2 = 0$$

where $d_{6,11}$ is the distance between particles 6 and 11. The first and second time derivatives of the cut-joints constraint equations lead to the corresponding velocity and acceleration equations respectively,

$$\dot{\mathbf{r}}_4 - \dot{\mathbf{r}}_{10} = 0 \text{ and } (\mathbf{r}_6 - \mathbf{r}_{11})^T (\dot{\mathbf{r}}_6 - \dot{\mathbf{r}}_{11}) = 0$$

$$\ddot{\mathbf{r}}_4 - \ddot{\mathbf{r}}_{10} = 0 \text{ and } (\mathbf{r}_6 - \mathbf{r}_{11})^T (\ddot{\mathbf{r}}_6 - \ddot{\mathbf{r}}_{11}) = -(\dot{\mathbf{r}}_6 - \dot{\mathbf{r}}_{11})^T (\dot{\mathbf{r}}_6 - \dot{\mathbf{r}}_{11})$$

The 4×8 matrix $\mathbf{C}$, defined in Eq. (4), can be written in the following closed form,

$$
\mathbf{C} = \begin{bmatrix}
0 & -\tilde{\mathbf{d}}_{10,1}\mathbf{u}_2 & -\tilde{\mathbf{d}}_{10,5} & \mathbf{u}_3 & 0 & 0 \\
0 & -\mathbf{d}_{6,11}^T \tilde{\mathbf{d}}_{6,1}\mathbf{u}_2 & -\mathbf{d}_{6,11}^T \tilde{\mathbf{d}}_{6,6} & 0 & -\mathbf{d}_{11,6}^T \mathbf{u}_4 & 0
\end{bmatrix}
$$

The kinematic constraints due to some common types of kinematic joints (e.g. revolute and spherical joints) can be automatically eliminated by properly locating the equivalent particles. The remaining kinematic constraints along with the geometric constraints are, in general, either linear or quadratic in the Cartesian coordinates of the particles. Therefore, the coefficients of their Jacobian matrix are constants or linear in the rectangular Cartesian coordinates. Where as in the formulation based on the relative coordinates, the constraint equations are derived based on loop closure equations which have the disadvantage that they do not directly determine the positions of the links and points of interest which makes the establishment of the dynamic problem more difficult. Also, the resulting constraint equations are highly nonlinear and contain complex circular functions. The absence of these circular functions in the point coordinate formulation leads to faster convergence and better accuracy. Furthermore, preprocessing the mechanism by the topological graph theory is not necessary as it would be the case with loop constraints. The additional manual work of the local axes attachment and local coordinates evaluation in the absolute coordinate formulation are not required which leads to fully computerized analysis. In addition, the number of constraint equations is much smaller

than the case of the absolute coordinate formulation which accounts for a reduction in the computational time and memory storage.

The resulting equations of motion, as given by Eq. (5), represent a 12×12 symmetric system of equations that can be solved for the unknown joint accelerations and the cut joints constraint forces. For the absolute coordinate formulation, a system of $30 + 27$ differential equations of motion plus algebraic equations of constraints is

Table 1 Time (s) variation of the vertical displacement (m) of the chassis using the suggested method and DAP-3D

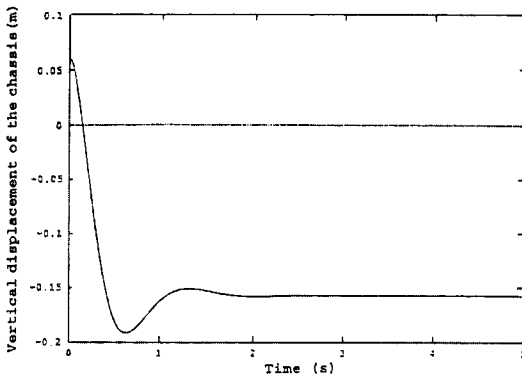| Simulation method	t=0.5	t=1	t=3
Suggested algorithm	−0.1786	−0.1571	−0.1536
DAP-3D	−0.1783	−0.1569	−0.1524

Fig. 2 The time (s) variation of the vertical displacement of the chassis (m)
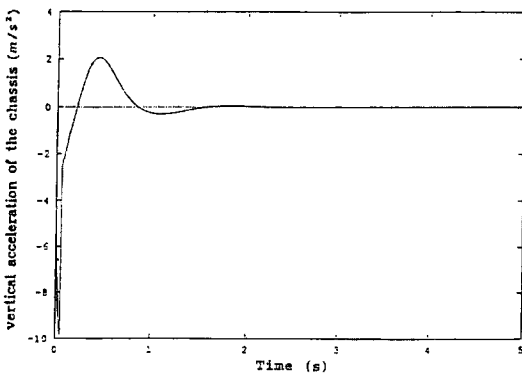

Fig. 3 The time (s) variation of the vertical displacement of the chassis (m/s^2)

constructed. Thus a resulting system of 57 differential-algebraic equations should be solved at every time step to determine the unknown accelerations and reaction forces. This reduction in the number of differential equations of motion and in turn the number of integrable variables is considered as an advantage for the point and joint coordinate formulation.

3.1 Results of the simulation

The above equations of motion, Eq. (5), are used to simulate the free response of the system from some initial condition where the initial Cartesian coordinates of the particles are given in Table A.4 in Appendix A. Figures 2 and 3 present the time variations of the vertical displacement and acceleration of the chassis respectively. The main chassis is accelerated downward due to the gravitational forces. Then, it undergoes a damped oscillatory motion controlled by the spring-damper-actuator elements forces and the wheels compression forces up to the steady state. The verification of the results is done by comparison with DAP-3D program which is based upon the absolute coordinates (Nikravesh and Attia, 1994). The comparison shows a complete agreement and coincidence in plotting for the results of the two simulations since the differences are not pronounced within the accuracy of the plotting. Also, it is more appropriate to present the results in tabular form. Table 1 compares the time variation of the vertical displacement of the chassis using the simulation results of the suggested algorithm and DAP-3D and ensures the validation of the results obtained here.

4. Conclusions

The application of a multibody dynamic formulation which uses both point and joint coordinates to study the dynamic analysis of the Macpherson strut suspension system is demonstrated in this paper. The Macpherson strut suspension is replaced by an equivalent system of 11 particles which results in a 34×34 constant sparse symmetric mass matrix. The equations of motion in terms of the Cartesian coordinates of the particles

are transformed to joint variables by defining suitable joint coordinates and using the velocity transformation matrix. The resulting differential equations in terms of the joint variables are equal to the number of degrees of freedom of the whole system plus the number of cut joints constraints. Use of both the Cartesian and relative joint variables produces an efficient set of equations without loss of generality. The results of the simulations are tested and compared with DAP-3D program. The chosen example incorporates open and closed chains with the common types of kinematic joints. The simulations indicate the simplicity and generality of the formulation.

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Appendix A

Table A.1 Description of the rigid bodies

Body Description	Mass (Kg)	Inertia (Kg·m²) $\xi\xi, \eta\eta, \zeta\zeta, \eta\zeta, \xi\zeta, \xi\eta$
Main chassis	456.0	570.0, 2320.0, 2715.0, 0.0, 0.0, 0.0
Lower arm	4.6	0.1, 0.1, 0.1, 0.0, 0.0, 0.0
Steering rod	12.1	0.25, 0.25, 0.25, 0.0, 0.0, 0.0
Tie rod	1.0	0.1, 0.1, 0.1, 0.0, 0.0, 0.0
Wheel	25.0	0.1, 0.1, 0.1, 0.0, 0.0, 0.0

Table A.2 The characteristics of the suspension springs and dampers

No.	Connected bodies	K (N/m)	D (Nsec/m)	l_o (m)
1	(1, 2)	4.22E+04	0.0E+00	0.4815
2	(1, 2)	0.0E+00	2.7E+03	0.0

Table A.3   The characteristics of the wheels

Radius	0.35 m
stiffness	2.00E+05 N/m
Damping Coefficient	8.0E+4 Nsee/m

Table A.4   Initial cartesian coordinates (mm) of the particles

$(x_1, y_1, z_1) = (270, -5, -10)$	$(x_7, y_7, z_7) = (452, -320, 20)$
$(x_2, y_2, z_2) = (540, -10, -20)$	$(x_8, y_8, z_8) = (452, -410, 20)$
$(x_3, y_3, z_3) = (570, 60, 90)$	$(x_9, y_9, z_9) = (456, -250, 246)$
$(x_4, y_4, z_4) = (460, -180, 470)$	$(x_{10}, y_{10}, z_{10}) = (460, -180, 470)$
$(x_5, y_5, z_5) = (450, -330, -40)$	$(x_{11}, y_{11}, z_{11}) = (470, 60, 90)$
$(x_6, y_6, z_6) = (560, -300, 60)$	

Appendix B

According to the chosen relative joint velocities, the linear velocities of the assigned points can be determined as follows: the velocity of point i $(i=1, \cdots, 4)$ on the chassis is given as,

$$\dot{\mathbf{r}}_i = \dot{\theta}_1 \mathbf{u}_1 \tag{B1}$$

The angular velocity of the A-arm is given by

$$\bar{\omega}_1 = \dot{\theta}_2 \mathbf{u}_2 \tag{B2}$$

Then, the velocity of point 5 is given as,

$$\dot{\mathbf{r}}_5 = \dot{\mathbf{r}}_1 + \bar{\omega}_1 X d_{5,1} = \dot{\theta}_1 \mathbf{u}_1 - \dot{\theta}_2 \tilde{\mathbf{d}}_{5,1} \mathbf{u}_2 \tag{B3}$$

The angular velocity of the Knuckle is given by,

$$\bar{\omega}_2 = \bar{\omega}_1 + \omega_1 \tag{B4}$$

The velocity of point j $(j=6, \cdots, 8)$ on the Knuckle is given as,

$$\dot{\mathbf{r}}_j = \dot{\mathbf{r}}_5 + \bar{\omega}_2 X \mathbf{d}_{j,5} = \dot{\theta}_1 \mathbf{u}_1 - \dot{\theta}_2 \tilde{\mathbf{d}}_{j,1} \mathbf{u}_2 - \tilde{\mathbf{d}}_{j,5} \omega_1 \tag{B5}$$

The velocity of point k $(k=9, 10)$ on the strut link is given as,

$$\dot{\mathbf{r}}_k = \dot{\mathbf{r}}_5 + \bar{\omega}_2 X \mathbf{d}_{k,5} \\ = \dot{\theta}_3 \mathbf{u}_3 - \dot{\theta}_1 \mathbf{u}_1 - \dot{\theta}_2 \tilde{\mathbf{d}}_{k,1} \mathbf{u}_2 - \tilde{\mathbf{d}}_{j,5} \omega_1 + \dot{\theta}_3 \mathbf{u}_3 \tag{B6}$$

The velocity of point 11 on the tie rod is given by,

$$\dot{\mathbf{r}}_{11} = \dot{\mathbf{r}}_3 + \dot{\theta}_4 \mathbf{u}_4 = \dot{\theta}_1 \mathbf{u}_1 + \dot{\theta}_4 \mathbf{u}_4 \tag{B7}$$